

hep-th/0609082
HMI-06-03
ITEP-TH-06-42
TCD-MATH-06-XX

Baxter Q -operator and Givental integral representation for C_n and D_n

A. Gerasimov

Institute for Theoretical and Experimental Physics, Moscow, 117259, Moscow, Russia
School of Mathematics, Trinity College, Dublin 2, Ireland
Hamilton Mathematics Institute, TCD, Dublin 2, Ireland,

D. Lebedev¹

Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia
l'Institut des Hautes Études Scientifiques, 35 route de Chartres, Bures-sur-Yvette, France,

S. Oblezin²

Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia
Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany,

Abstract

Recently integral representations for the eigenfunctions of quadratic open Toda chain Hamiltonians for classical groups was proposed. This representation generalizes Givental representation for A_n . In this note we verify that the wave functions defined by these integral representations are common eigenfunctions for the complete set of open Toda chain Hamiltonians. We consider the zero eigenvalue wave functions for classical groups C_n and D_n thus completing the generalization of the Givental construction in these cases. The construction is based on a recursive procedure and uses the formalism of Baxter Q -operators. We also verify that the integral Q -operators for C_n , D_n and twisted affine algebra $A_{2n-1}^{(2)}$ proposed previously intertwine complete sets of Hamiltonian operators. Finally we provide integral representations of the eigenfunctions of the quadratic D_n Toda chain Hamiltonians for generic nonzero eigenvalues.

¹E-mail: lebedev@ihes.fr

²E-mail: Sergey.Oblezin@itep.ru

1 Introduction

In [GLO] we provided integral representations for zero eigenvalue wave functions of open Toda chain quadratic Hamiltonian operators corresponding to the classical series of finite Lie algebras B_n , C_n , and D_n . This representation generalizes Givental representation for A_n [Gi]. In this short note we verify the eigenvalue properties for the full set of the Hamiltonian operators for C_n and D_n open Toda chains. We also give as an example of the integral representations for generic eigenvalues the explicit expression for the wave function of D_n Toda chain.

We use a recursive representation of the common eigenfunctions of the full sets of C_n and D_n Toda chain Hamiltonians. The integral recursive operator is naturally represented as a composition of two elementary integral operators with simple integral kernels. The interesting property of these elementary integral operators is that they relate C_n and D_n wave functions. This is the main reason why we consider these two cases together.

Baxter Q -operator is a key tool in the theory of quantum integrable systems [B]. It plays the role of the generator of quantum Backlund transformations and was explicitly constructed for $A_n^{(1)}$ closed Toda chain in [PG]. It was noted in [GKLO] that the recursive operators for A_n open Toda chain obtained as a limit of Q -operators of $A_n^{(1)}$ Toda chain play the crucial role in the construction of the Givental integral representation. In [GLO] we generalize this relation by constructing $A_{2n-1}^{(2)}$ Baxter Q -operator and demonstrating that C_n and D_n recursive operators can be obtained from $A_{2n-1}^{(2)}$ Baxter Q -operator in a similar limit. The properties of the recursive operators are then follows from the commutation relation between quantum L -operator and Q -operator of $A_{2n-1}^{(2)}$ closed Toda chain. In this note we verify that the integral Q -operators for $A_{2n-1}^{(2)}$ Toda chain introduced in [GLO] commute with the full set of $A_{2n-1}^{(2)}$ Toda chain Hamiltonians. The proof is based on the explicit commutation relations with L -operator for Toda chain. In a certain limit this leads to the commutation relations between recursive operators and Hamiltonian operators of C_n and D_n open Toda chains. The fact that the wave functions constructed in [GLO] are common eigenfunctions of the full set of the Toda chain then easily follows from the commutation relations. The use of $A_{2n-1}^{(2)}$ is not very essential in this construction and is dictated by the simplicity of the resulting degeneration procedure. For example similar relations exist between Q -operators for $C_n^{(1)}$ and $D_n^{(1)}$ and recursive operators for C_n and D_n . These constructions will be presented elsewhere together with a complete account of the constructions of the Baxter Q -operators, recursive operators and integral representations of wave functions for all classical series [GLO1].

The plan of this note is as follows. In Section 2 we recall explicit expressions for the Baxter Q -operator for $A_{2n-1}^{(2)}$ given in [GLO] and prove that it commutes with the full set of $A_{2n-1}^{(2)}$ Toda chain Hamiltonian operators. In Section 3 we derive recursive operators for C_n and D_n as a certain limit of the $A_{2n-1}^{(2)}$ Baxter Q -operator. We prove the intertwining relations of the recursive operators with Hamiltonians of C_n and D_n Toda chains. In Section 4 integral representations of the eigenfunctions of D_n Toda chain quadratic Hamiltonian operators for generic eigenvalues are given.

Finally in the Appendix we explicitly check that thus obtained wave functions for D_2 open

Toda chain can be reduced to a product of the wave function of two independent A_1 Toda chains. Although it is a consequence of the isomorphism $D_2 = A_1 \oplus A_1$ the transformation is rather nontrivial and serves as an independent check of our construction.

Acknowledgments: The research of A. Gerasimov is partly supported by the grant RFBR 04-01-00646 and Enterprise Ireland Basic Research Grant; The research of D. Lebedev is partially supported by the grant RFBR 04-01-00646. The research of S. Oblezin was partially supported by the grant RFBR 04-01-00642. D. Lebedev is grateful to Institute des Hautes Études Scientifiques for a warm hospitality. S. Oblezin is grateful to Max-Planck-Institut für Mathematik for excellent working conditions.

2 Baxter Q -operator for $A_{2n-1}^{(2)}$

In this section we recall the construction of Q -operator for $A_{2n-1}^{(2)}$ and explicitly check that it commutes with the full set of $A_{2n-1}^{(2)}$ closed Toda chain Hamiltonians. Let us start with the description of $A_{2n-1}^{(2)}$ closed Toda chain. For the relevant facts on the root systems see [K],[DS]. The detailed description of the corresponding Toda chains can be found in [RSTS].

Let us fix an orthonormal basis $\{e_1, \dots, e_n\}$ in \mathbb{R}^n . Simple roots of the twisted affine root system $A_{2n-1}^{(2)}$ can be represented in the following form

$$\alpha_1 = 2e_1, \quad \alpha_{i+1} = e_{i+1} - e_i, \quad 1 \leq i \leq n-1 \quad \alpha_{n+1} = -e_n - e_{n-1}, \quad (2.1)$$

The corresponding Dynkin diagram is given by

$$\begin{array}{ccccccc} \alpha_1 & \Rightarrow & \alpha_2 & \longrightarrow & \dots & \longrightarrow & \alpha_{n-1} & \longrightarrow & \alpha_n \\ & & & & & & \downarrow & & \\ & & & & & & \alpha_{n+1} & & \end{array}$$

To construct L -operator of the closed Toda chain corresponding to an affine root system one should choose an evaluation representation of the corresponding affine Lie algebra. Let $\{e_i, h_i, f_i\}$ be a bases of the twisted Lie algebra \mathfrak{g} corresponding to root system $A_{2n-1}^{(2)}$. We define the evaluation representation of $\widehat{\mathfrak{g}}$ as follows (see e.g. [DS]). Consider the following homomorphism

$$\pi : \widehat{\mathfrak{g}} \longrightarrow \text{Mat}(2n, \mathbb{C}) \otimes \mathbb{C}[u, u^{-1}] \quad (2.2)$$

defined explicitly as

$$\begin{aligned} \pi(e_0) &= u(E_{1,2n-1} + E_{2,2n})/2, & \pi(f_0) &= 2u^{-1}(E_{2n-1,1} + E_{2n,2}), \\ \pi(e_i) &= E_{i+1,i} + E_{2n+1-i,2n-i}, & \pi(f_i) &= E_{i,i+1} + E_{2n-i,2n+1-i}, \\ \pi(e_n) &= E_{n+1,n}, & \pi(f_n) &= 2E_{n,n+1}, \\ \pi(h_0) &= (E_{1,1} - E_{2n,2n}) + (E_{2,2} - E_{2n-1,2n-1}), \\ \pi(h_i) &= (E_{i+1,i+1} - E_{i,i}) + (E_{2n+1-i,2n+1-i} - E_{2n-i,2n-i}), \\ \pi(h_n) &= 2(E_{n+1,n+1} - E_{n,n}) \end{aligned} \quad (2.3)$$

where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ in a standard bases in $Mat(2n, \mathbb{C})$. The evaluation homomorphism then defined by the evaluation at a particular value of the variable u .

Then classical L -operator for $A_{2n-1}^{(2)}$ Toda chain has the following form in this representation

$$L(p, x; u) = \sum_{i=1}^n p_{x_i} \left(E_{i,i} - E_{2n+1-i, 2n+1-i} \right) + q_i(x) \left(E_{i,i+1} - E_{2n-i, 2n+1-i} \right) - \left(E_{i+1,i} - E_{2n+1-i, 2n-i} \right) + 2q_0(x) \left(E_{2n-1,1} - E_{2n,2} \right) - \frac{u}{2} \left(E_{1,2n-1} - E_{2,2n} \right), \quad (2.4)$$

where $q_i(x) = e^{(\alpha_i, x)}$ and $x = \sum_{i=1}^n x_i e_i$. The space of L -operators (2.4) provides a model for the phase space of the classical $A_{2n-1}^{(2)}$ Toda chain. The quantum \mathcal{L} -operator is obtained by the standard substitution $p_{x_j} = -i\hbar \partial_{x_j}$.

$$\mathcal{L}(\partial_x, x; u) = \sum_{i=1}^n -i \left(E_{i,i} - E_{2n+1-i, 2n+1-i} \right) \frac{\partial}{\partial x_i} + q_i(x) \left(E_{i,i+1} - E_{2n-i, 2n+1-i} \right) - \left(E_{i+1,i} - E_{2n+1-i, 2n-i} \right) + 2q_0(x) \left(E_{2n-1,1} - E_{2n,2} \right) - \frac{u}{2} \left(E_{1,2n-1} - E_{2,2n} \right). \quad (2.5)$$

Classical Hamiltonian operators of the Toda chain are given by the coefficients of the characteristic polynomial of L

$$\det(L(u) - \lambda) = u + \frac{1}{u} + \sum_{i=k}^N h_{2k}(p, x) \lambda^{2n-2k} \quad (2.6)$$

For example the quadratic Hamiltonian generator is given by

$$H_2^{A_{2n-1}^{(2)}}(x) = \frac{1}{2} \sum_{i=1}^n p_{x_i}^2 + e^{2x_1} + \sum_{i=1}^{n-1} e^{x_{i+1}-x_i} + e^{-x_n-x_{n-1}} \quad (2.7)$$

In [GLO] we introduced the following integral operator defined by the integral kernel

$$\begin{aligned} Q_{A_{2n-1}^{(2)}}(x_i, z_i) &= \exp\left(\frac{i}{\hbar} \mathcal{F}(x_i, z_i)\right) = \\ &= \exp \frac{i}{\hbar} \left\{ g_1 e^{x_1+z_1} + \sum_{i=1}^{n-1} \left(e^{x_i-z_i} + g_{i+1} e^{z_{i+1}-x_i} \right) + e^{x_n-z_n} + g_{n+1} e^{-x_n-z_n} \right\}, \end{aligned} \quad (2.8)$$

The action of the Q -operator on the functions is given by

$$(Q_{A_{2n-1}^{(2)}} f)(x_i) = \int \bigwedge_{i=1}^n dz_i Q_{A_{2n-1}^{(2)}}(x_i, z_i) f(z_i) \quad (2.9)$$

Note that \mathcal{F} is a generating function of the canonical transformation corresponding to the integral operator in the classical limit.

It was shown in [GLO] that thus defined integral operator (2.8) intertwines quadratic Hamiltonian operators $H_2^{A_{2n-1}^{(2)}}$ for different coupling constants

$$H_2(x_i, \partial_{x_i}, g_i) Q(x_i, z_i) = Q(x_i, z_i, g_i) H_2(z_i, \partial_{z_i}, g'_i) \quad (2.10)$$

where we use the quadratic Hamiltonian operators

$$H_2^{A_{2n-1}^{(2)}}(x) = \frac{1}{2} \sum_{i=1}^n p_{x_i}^2 + 2g_1 e^{2x_1} + \sum_{i=1}^{n-1} g_i e^{x_{i+1}-x_i} + g_n g_{n+1} e^{-x_n-x_{n+1}} \quad (2.11)$$

and the relation holds for $g_i = g'_{n+2-i} = 1$, $i \neq 1$ and $g_1 = g'_{n+1} = 2$ (see [GLO] for details). In the following we will assume that these particular coupling constants are chosen. Also note that in (2.10) and in the following similar identities we shall assume that the Hamiltonian operator on l.h.s. acts on the right and the Hamiltonian on r.h.s. acts on the left.

Let us show that the integral operator Q commutes with all Hamiltonian operators of $A_{2n-1}^{(2)}$ Toda chain. The prove is based on the following result.

Theorem 2.1 *Let $\mathcal{L}(x_i, \partial_{x_i}, g_i, u)$ be a quantum L -operator given by (2.5) and $Q(x_i, z_i)$ be an integral kernel (2.8). Then the following relation holds*

$$R(x_i, z_i, g_i, g'_i, u) \mathcal{L}(x_i, \partial_{x_i}, g_i, u) Q(x_i, z_i, u) = Q(x_i, z_i, u) \mathcal{L}(z_i, \partial_{z_i}, g'_i, u) R(x_i, z_i, g_i, g'_i, u)$$

where matrix R is given by

$$\begin{aligned} R(x_i, z_i, g_i, g'_i, u) = & \left(e^{x_1+z_1} E_{n,1} + \frac{u}{2} E_{1,n} \right) + \sum_{i=0}^n \left(e^{z_i-x_{i-1}} E_{n+1,i+1} - u E_{i,n+i} \right) + \\ & \left(e^{-x_n-z_n} E_{2n,n+1} + \frac{u}{2} E_{n+1,2n} \right) + \sum_{i=1}^n \left(e^{x_{n+1-i}-z_{n+1-i}} E_{i,n+i} - E_{n+i,i} \right) \end{aligned} \quad (2.12)$$

The proof is straightforward and reduces to the verification of the matrix identity

$$R(x_i, z_i, g_i, g'_i, u) L(x_i, p_{x_i}, g_i, u) = L(z_i, p_{z_i}, g'_i, u) R(x_i, z_i, g_i, g'_i, u) \quad (2.13)$$

where

$$p_{x_1} = \frac{\partial \mathcal{F}}{\partial x_1} = e^{z_1+x_1} + e^{x_1-z_1} - e^{z_2-x_1}, \quad p_{x_i} = \frac{\partial \mathcal{F}}{\partial x_i} = e^{x_i-z_i} - e^{z_{i+1}-x_i}, \quad (2.14)$$

$$p_{z_i} = -\frac{\partial \mathcal{F}}{\partial z_i} = e^{x_i-z_i} - e^{z_i-x_{i-1}}, \quad p_{z_n} = -\frac{\partial \mathcal{F}}{\partial z_n} = e^{-x_n-z_n} + e^{x_n-z_n} - e^{z_n-x_{n+1}} \quad (2.15)$$

Note that the identity (2.13) follows from the following representations of the pair of L -operators

$$L(x_i, p_{x_i}, g_i, u) = R R^*, \quad L(z_i, p_{z_i}, g'_i, u) = R^* R, \quad (2.16)$$

where R is given by (2.12) and relations (2.14) are implied. The involution $R \rightarrow R^*$ is the involution that describes the twisted affine Lie algebra $A_{2n-1}^{(2)}$ as a fixed point subalgebra of $A_{2n-1}^{(1)}$ (see [DS] for the details). The equations (2.16) implies that L is in the image of $A_{2n-1}^{(2)}$.

We would like also note that the similar type of relations was used in the theory of Darboux maps (see [AvM],[V]).

Example 2.1 *In the case of $n = 4$ adopt the following notations*

$$a_i = e^{x_i - z_i}, \quad d_0 = e^{x_1 + z_1}, \quad d_i = e^{z_i - x_{i-1}}, \quad d_n = e^{-x_n - z_n}.$$

In these terms we get for $L(x_i, p_{x_i}, g_i, u)$ with $g_i = 1$

$$\begin{pmatrix} a_4 - d_4 & a_4 d_3 & 0 & 0 & 0 & 0 & -u/2 & 0 \\ -1 & a_3 - d_3 & a_3 d_2 & 0 & 0 & 0 & 0 & -u/2 \\ 0 & -1 & a_2 - d_2 & a_2 d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & a_1 + d_0 - d_1 & 2a_1 d_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -a_1 - d_0 + d_1 & 2a_2 d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -a_2 + d_2 & a_3 d_2 & 0 \\ 2u^{-1} d_3 d_4 & 0 & 0 & 0 & 0 & -1 & -a_3 + d_3 & a_4 d_3 \\ 0 & 2u^{-1} d_3 d_4 & 0 & 0 & 0 & 0 & -1 & -a_4 + d_4 \end{pmatrix}.$$

Then we have

$$R = \begin{pmatrix} 0 & 0 & 0 & -u/2 & e^{x_4 - z_4} u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u & e^{x_3 - z_3} u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -u & e^{x_2 - z_2} u & 0 \\ 2e^{z_1 + x_1} & 0 & 0 & 0 & 0 & 0 & -u & e^{x_1 - z_1} u \\ -1 & e^{z_2 - x_1} & 0 & 0 & 0 & 0 & 0 & -u/2 \\ 0 & -1 & e^{z_3 - x_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & e^{z_4 - x_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2e^{-z_4 - x_4} & 0 & 0 & 0 \end{pmatrix} \quad (2.17)$$

and

$$R^* = \begin{pmatrix} 0 & 0 & 0 & 1/2 & e^{x_1 - z_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & e^{x_2 - z_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & e^{x_3 - z_3} & 0 \\ 2u^{-1} e^{-z_4 - x_4} & 0 & 0 & 0 & 0 & 0 & 1 & e^{x_4 - z_4} \\ u^{-1} & u^{-1} e^{z_4 - x_3} & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & u^{-1} & u^{-1} e^{z_3 - x_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u^{-1} & u^{-1} e^{z_2 - x_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^{-1} & 2u^{-1} e^{z_1 + x_1} & 0 & 0 & 0 \end{pmatrix}$$

Taking into account (2.13) we have the following proposition.

Proposition 2.1 *The following identity holds*

$$\det(L(x_i, p_{x_i}, g_i, u) - \lambda) = \det(L(z_i, p_{z_i}, g'_i, u) - \lambda) \quad (2.18)$$

The same reasoning goes for the quantum L -operators. Let $H_i^{A_{2n-1}^{(2)}}(x_i, \partial_{x_i}, g_i)$ be a full set of the commuting quantum Hamiltonian operators for $A_{2n-1}^{(2)}$ Toda chain given by the quantization of the classical Hamiltonian operators (2.6). The quantization uses the recursive representation for the characteristic polynomial of the quantum L -operator and is based on the special properties of the matrix differential operator (2.5). Then the following intertwining relations holds

$$H_i^{A_{2n-1}^{(2)}}(x_i, \partial_{x_i}, g_i)Q(x_i, z_i) = Q(x_i, z_i)H_i^{A_{2n-1}^{(2)}}(z_i, \partial_{z_i}, g'_i)$$

Let us stress that the constructed integral operators intertwine the Hamiltonians of the $A_{2n-1}^{(2)}$ Toda chain with different coupling constants. It was explained in [GLO] that to get the Baxter Q -operator for $A_{2n-1}^{(2)}$ commuting with the Hamiltonians one should apply the elementary intertwining twice

$$Q^{A_{2n-1}^{(2)}}(x_1, \dots, x_n; y_1, \dots, y_n) = \int \bigwedge_{i=1}^n dz_i Q(x_1, \dots, x_n; z_1, \dots, z_n) \times \quad (2.19)$$

$$\times Q(y_1, \dots, y_n; z_1, \dots, z_n).$$

3 Recursive operators for C_n and D_n

In this section we prove the intertwining relations between recursive operators for the full set of Hamiltonian operators for C_n and D_n open Toda chains. These commutation relations can be also obtained from the analogous properties of Baxter Q -operator of $A_{2n-1}^{(2)}$ Toda chain in a certain limit.

We start with the description of C_n and D_n Toda chains. The root system C_n and D_n and the Hamiltonian are defined in terms of the orthonormal basis $\{e_1, \dots, e_n\}$ in \mathbb{R}^n as follows. Simple roots of the root system C_n are given by

$$\alpha_1 = 2e_1, \quad \alpha_{i+1} = e_{i+1} - e_i, \quad 1 \leq i \leq n-1 \quad \alpha_{n+1} = -e_n - e_{n-1}, \quad (3.1)$$

The corresponding Dynkin diagram is

$$\begin{array}{ccccccc} \alpha_1 & \Rightarrow & \alpha_2 & \longrightarrow & \dots & \longrightarrow & \alpha_{n-1} & \longrightarrow & \alpha_n \\ & & & & & & \downarrow & & \\ & & & & & & \alpha_{n+1} & & \end{array}$$

Simple roots of the root system D_n are given by

$$\alpha_1 = 2e_1, \quad \alpha_{i+1} = e_{i+1} - e_i, \quad 1 \leq i \leq n-1 \quad \alpha_{n+1} = -e_n - e_{n-1}, \quad (3.2)$$

The corresponding Dynkin diagram is

$$\begin{array}{ccccccc} \alpha_1 & \Rightarrow & \alpha_2 & \longrightarrow & \dots & \longrightarrow & \alpha_{n-1} & \longrightarrow & \alpha_n \\ & & & & & & \downarrow & & \\ & & & & & & \alpha_{n+1} & & \end{array}$$

Classical Hamiltonian operators $h_i^{C_n}$ of C_n Toda chain are given by the coefficients of the characteristic polynomial

$$\det(L(x, p) - \lambda) = \sum_i h_i \lambda^{n-i} \quad (3.3)$$

where L operator is given by

$$\begin{aligned} L^{C_n}(x, p) = & \sum_{i=1}^n p_i \left(E_{i,i} - E_{2n+1-i, 2n+1-i} \right) + \\ & q_i(x) \left(E_{i,i+1} + E_{2n-i, 2n+1-i} \right) - \left(E_{i+1,i} - E_{2n+1-i, 2n-i} \right) \end{aligned} \quad (3.4)$$

Similarly classical Hamiltonian operators of D_n Toda chain are given by the coefficients of characteristic polynomial (3.3) of the following L -operator

$$\begin{aligned} L^{D_n}(x, p) = & \sum_{i=1}^n p_i \left(E_{i,i} - E_{2n+1-i, 2n+1-i} \right) + \\ & \sum_{i=1}^{n-1} q_i(x) \left(E_{i,i+1} + E_{2n-i, 2n+1-i} \right) - \left(E_{i+1,i} - E_{2n+1-i, 2n-i} \right) + \\ & 2q_n(x) \left(E_{n-1, n+1} + E_{n, n+2} \right) - (E_{n+1, n-1} + E_{n+2, n}) \end{aligned} \quad (3.5)$$

The corresponding quantum Hamiltonians are obtained by the standard quantisation procedure. Thus for the quadratic Hamiltonian operators we have

$$\begin{aligned} H_2^{C_n}(z_i) = & -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2} + \sum_{i=1}^{n-1} e^{z_{i+1}-z_i} + e^{-2z_n}, \\ H_2^{D_n}(x_i) = & -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n-1} e^{x_{i+1}-x_i} + e^{-x_n-x_{n-1}} \end{aligned} \quad (3.6)$$

In [GLO] the integral operators intertwining quadratic Toda chain Hamiltonian operators for different elements of the classical series C_n and D_n were proposed. Integral operator defined by the kernel

$$Q_{C_n}^{D_n}(x_i; z_i) = \exp \frac{i}{\hbar} \mathcal{F}_{C_n}^{D_n}(x_i; z_i) = \exp \frac{i}{\hbar} \left\{ \sum_{i=1}^{n-1} \left(e^{x_i-z_i} + e^{z_{i+1}-x_i} \right) + e^{x_n-z_n} + e^{-x_n-z_n} \right\}, \quad (3.7)$$

satisfies the following intertwining relation

$$H_2^{C_n}(x_i, \partial_{x_i}) Q_{D_n}^{C_n}(x_i, z_i) = Q_{D_n}^{C_n}(x_i, z_i) H_2^{D_n}(z_i, \partial_{z_i}). \quad (3.8)$$

Similarly integral operator defined by the following kernel

$$Q_{D_n}^{C_{n-1}}(x_i; z_i) = \exp \frac{i}{\hbar} \mathcal{F}_{D_n}^{C_{n-1}}(x_i; z_i) = \exp \frac{i}{\hbar} \left\{ \sum_{i=1}^{n-1} \left(e^{z_i-x_i} + g_i e^{x_{i+1}-z_i} \right) + g_n e^{-x_n-z_{n-1}} \right\}, \quad (3.9)$$

satisfies the following intertwining relation

$$H_2^{D_n}(x_i, \partial_{x_i}) Q_{D_n}^{C_{n-1}}(x_i, z_i) = Q_{D_n}^{C_{n-1}}(x_i, z_i) H_2^{C_{n-1}}(z_i, \partial_{z_i}) \quad (3.10)$$

In the classical approximation (3.8) is reduced to the identity

$$H_2^{C_n}(x_i, p_{x_i}) = H_2^{D_n}(z_i, p_{z_i}) \quad (3.11)$$

where the following expressions for the momenta variables are implied

$$\begin{aligned} p_{x_i} &= \frac{\partial \mathcal{F}_{C_n}^{D_n}}{\partial x_i} = e^{x_i - z_i} - e^{z_{i+1} - x_i}, \\ p_{x_n} &= \frac{\partial \mathcal{F}_{C_n}^{D_n}}{\partial x_n} = e^{x_n - z_n} - e^{-z_n - x_n} \quad 1 \leq i < n, \end{aligned} \quad (3.12)$$

$$\begin{aligned} p_{z_1} &= -\frac{\partial \mathcal{F}_{C_n}^{D_n}}{\partial z_1} = e^{x_1 - z_1}, \quad p_{z_i} = -\frac{\partial \mathcal{F}_{C_n}^{D_n}}{\partial z_i} = e^{x_i - z_i} - e^{z_i - x_{i-1}}, \\ p_{z_n} &= -\frac{\partial \mathcal{F}_{C_n}^{D_n}}{\partial z_n} = e^{x_n - z_n} + e^{-z_n - x_n} - e^{z_n - x_{n-1}}, \quad 1 < i < n. \end{aligned} \quad (3.13)$$

Similarly (3.10) is reduced in the classical approximation to

$$H_2^{D_n}(x_i, p_{x_i}) = H_2^{C_{n-1}}(z_i, p_{z_i}) \quad (3.14)$$

where the following expressions for the momenta variables are implied

$$\begin{aligned} p_{x_1} &= \frac{\partial \mathcal{F}_{D_n}^{C_{n-1}}}{\partial x_1} = -e^{z_1 - x_1}, \quad p_{x_i} = \frac{\partial \mathcal{F}_{D_n}^{C_{n-1}}}{\partial x_i} = e^{x_i - z_{i-1}} - e^{z_i - x_i}, \\ p_{x_n} &= \frac{\partial \mathcal{F}_{D_n}^{C_{n-1}}}{\partial x_n} = e^{x_n - z_{n-1}} - e^{-z_{n-1} - x_n}, \quad 1 < i < n, \\ p_{z_i} &= -\frac{\partial \mathcal{F}_{D_n}^{C_{n-1}}}{\partial z_i} = e^{x_{i+1} - z_i} - e^{z_i - x_i}, \\ p_{z_{n-1}} &= \frac{\partial \mathcal{F}_{D_n}^{C_{n-1}}}{\partial z_{n-1}} = e^{-z_{n-1} - x_n} + e^{x_n - z_{n-1}} - e^{z_{n-1} - x_{n-1}}, \quad 1 \leq i < n-1. \end{aligned} \quad (3.15)$$

Toda chains corresponding to the root systems C_n and D_n can be obtained from $A_{2n-1}^{(2)}$ Toda chain using an appropriate limiting procedure. To do this explicitly one should explicitly introduce the coupling constant in Toda chains. It was shown in [GLO] that the operator

$$\begin{aligned} Q_{A_{2n-1}^{(2)}}(x_i, z_i) &= \frac{i}{\hbar} \exp \mathcal{F}(x_i, z_i) = \\ &= \exp \frac{i}{\hbar} \left\{ g_1 e^{x_1 + z_1} + \sum_{i=1}^{n-1} \left(e^{x_i - z_i} + g_{i+1} e^{z_{i+1} - x_i} \right) + e^{x_n - z_n} + g_{n+1} e^{-x_n - z_n} \right\} \end{aligned} \quad (3.17)$$

satisfies the following relation

$$H_2^{A_{2n-1}^{(2)}}(x_i)Q_{A_{2n-1}^{(2)}}(x_i, z_i) = Q_{A_{2n-1}^{(2)}}(x_i, z_i)\tilde{H}_2^{A_{2n-1}^{(2)}}(z_i) \quad (3.18)$$

for the quadratic Hamiltonians associated to the root system of type $A_{2n-1}^{(2)}$:

$$H_2^{A_{2n-1}^{(2)}}(x_i) = -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 2g_1 e^{2x_1} + \sum_{i=1}^{n-1} g_{i+1} e^{x_{i+1}-x_i} + g_n g_{n+1} e^{-x_n-x_{n-1}} \quad (3.19)$$

and

$$\tilde{H}_2^{A_{2n-1}^{(2)}}(z_i) = -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2} + g_1 g_2 e^{z_1+z_2} + \sum_{i=1}^{n-1} g_{i+1} e^{z_{i+1}-z_i} + 2g_{n+1} e^{-2z_n} \quad (3.20)$$

Taking the limit $g_1 \rightarrow 0$ we obtain that Q -operator

$$Q_{C_n}^{D_n}(x_i; z_i) = \lim_{g_1 \rightarrow 0} Q_{A_{2n-1}^{(2)}}(x_i, z_i) \quad (3.21)$$

intertwines

$$H_2^{C_n}(z_i) = \lim_{g_1 \rightarrow 0} \tilde{H}_2^{A_{2n-1}^{(2)}}(z_i) \quad (3.22)$$

and

$$H_2^{D_n}(x_i) = \lim_{g_1 \rightarrow 0} H_2^{A_{2n-1}^{(2)}}(x_i). \quad (3.23)$$

These Hamiltonians are naturally associated with root systems of type C_n and D_n respectively that are isomorphic to certain root sub-systems in the affine root system of type $A_{2n-1}^{(2)}$. Thus this limit procedure provides the L -operators associated to the root sub-systems of types C_n and D_n .

Note that the limiting operator $Q_{C_n}^{D_n}(x_i; z_i)$ in general intertwines Hamiltonians of Toda chains with different coupling constants except last ones. Thus the following operators :

$$H_2^{C_n}(z_i) = -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2} + \sum_{i=1}^{n-1} g'_{i+1} e^{z_{i+1}-z_i} + 2g_n e^{-2z_n} \quad (3.24)$$

and

$$H_2^{D_n}(x_i) = -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n-1} g_{i+1} e^{x_{i+1}-x_i} + g_{n-1} g_n e^{-x_n-x_{n-1}} \quad (3.25)$$

are intertwined by the integral operator with the kernel

$$Q_{C_n}^{D_n}(x_i; z_i) = \exp \left\{ \sum_{i=1}^{n-1} \left(\gamma_i e^{x_i-z_i} + \beta_i e^{z_{i+1}-x_i} \right) + e^{x_n-z_n} + g_n e^{-x_n-z_n} \right\}$$

where

$$\gamma_i = \prod_{k=i}^{n-1} \frac{g'_k}{g_k}, \quad \beta_i = g_i \prod_{k=i+1}^{n-1} \frac{g_k}{g'_k} \quad 1 \leq i < n. \quad (3.26)$$

Taking the limit $g'_1 \rightarrow 0$ we obtain

$$H_2^{C_{n-1}}(z_i) = \lim_{g'_1 \rightarrow 0} H_2^{C_n}(z_i) \quad (3.27)$$

Thus the intertwiner

$$Q_{C_{n-1}}^{D_n}(x_i; z_i) = \lim_{g'_1 \rightarrow 0} Q_{C_n}^{D_n}(x_i; z_i) \quad (3.28)$$

satisfying

$$H_2^{D_n}(x_i) Q_{C_{n-1}}^{D_n}(x_i, z_i) = Q_{C_{n-1}}^{D_n}(x_i, z_i) H_2^{C_{n-1}}(z_i) \quad (3.29)$$

Therefore all the recursive operators can be obtained from $A_{2n-1}^{(2)}$ Baxter Q -operators in different limits. This implies that the generalization of the intertwining relations (3.8) and (3.10) to the full set of the Hamiltonians of C_n and D_n open Toda chains can be obtained from (2.18), (2.19) using the limiting procedure described above. The following commutation relations with L operators are the analog of the Theorem 2.1.

Theorem 3.1 *Let $\mathcal{L}^{C_n}(x_i, \partial_{x_i}, g_i, u)$ be a quantum L -operator of C_n Toda chain and let $\mathcal{L}^{D_n}(x_i, \partial_{x_i}, g_i, u)$ be a quantum L -operator of D_n Toda chain. Then the following intertwining relations for the kernels (3.7), (3.9) of the integral operators hold*

$$\begin{aligned} M(x_i, z_i, g_i, g'_i, u) \mathcal{L}^{C_n}(x_i, \partial_{x_i}, g_i, u) Q_{C_n}^{D_n}(x_i, z_i, u) = \\ = Q_{C_n}^{D_n}(x_i, z_i) \mathcal{L}^{D_n}(z_i, \partial_{z_i}, g'_i, u) M(x_i, z_i, g_i, g'_i, u) \end{aligned} \quad (3.30)$$

$$\begin{aligned} N(x_i, z_i, g_i, g'_i, u) \mathcal{L}^{D_n}(x_i, \partial_{x_i}, g_i, u) Q_{D_n}^{C_{n-1}}(x_i, z_i, u) = \\ = Q_{D_n}^{C_{n-1}}(x_i, z_i) \mathcal{L}^{C_{n-1}}(z_i, \partial_{z_i}, g'_i, u) N(x_i, z_i, g_i, g'_i, u) \end{aligned} \quad (3.31)$$

where matrices M and N are given

$$\begin{aligned} M(x_i, z_i, g_i, g'_i, u) = \sum_{i=1}^n \left(E_{i,i} + e^{z_i - x_{i-1}} E_{i,i+1} \right) + \\ + \left(e^{x_{n+1-i} - z_{n+1-i}} E_{2n+1-i, 2n+1-i} + E_{2n+1-i, 2n-i} \right) + \left(e^{-x_n - z_n} E_{n, n+1} - \frac{1}{2} E_{n+1, n} \right) \end{aligned} \quad (3.32)$$

$$\begin{aligned} N(x_i, z_i, g_i, g'_i, u) = \sum_{i=1}^n \left(e^{x_i - z_i} E_{i,i} - E_{i+1, i} \right) \\ + \left(E_{2n+1-i, 2n+1-i} - e^{z_{n+1-i} - z_{n-i}} E_{2n-i, 2n+1-i} \right) + \left(e^{-x_n - z_n} E_{n, n+1} + \frac{1}{2} E_{n+1, n} \right) \end{aligned} \quad (3.33)$$

Similar to the case of $A_{2n-1}^{(2)}$ the commutation relations (3.30), (3.31) provide the quantum intertwining relation with C_n and D_n Hamiltonian operators. This leads to the proof of the zero eigenvalue property for the full sets of the Hamiltonian operators applied to the corresponding wave functions [GLO].

4 Integral representation of generic D_n eigenfunctions

In [GLO] integral representations for zero eigenvalue wave functions of D_n Toda chain quadratic Hamiltonian operators were given. In this Section we generalize these representations to the case of generic eigenvalues.

Following [GLO] we represent the wave functions of D_n Toda chain using the recursive integral operators. Let us define an integral recursive operator by the following kernel

$$\begin{aligned} Q_{D_{k+1}}^{D_k}(x_{k+1,i}, x_{k,j}, \lambda_{k+1}) &= \int \bigwedge_{i=1}^k dz_{k,i} \exp \left\{ i\lambda_{k+1} \left(\sum_{i=1}^{k+1} x_{k+1,i} + \sum_{i=1}^k x_{k,i} - 2 \sum_{i=1}^k z_{k,i} \right) \right\} \times \\ &\quad \times \left(e^{-x_{k+1,k+1}} + e^{-x_{k,k}} \right)^{2i\lambda_{k+1}} \times \\ &\quad \times Q_{D_{k+1}}^{C_k}(x_{k+1,1}, \dots, x_{k+1,k+1}; z_{k,1}, \dots, z_{k,k}) Q_{C_k}^{D_k}(z_{k,1}, \dots, z_{k,k}; x_{k,1}, \dots, x_{k,k}), \end{aligned} \quad (4.1)$$

where

$$Q_{D_{k+1}}^{C_k}(x_{k+1,i}, z_{k,i}) = \exp \left\{ \sum_{i=1}^k \left(e^{z_{k,i} - x_{k+1,i}} + e^{x_{k+1,i+1} - z_{k,i}} \right) + e^{-x_{k+1,k+1} - z_{k,k}} \right\}, \quad (4.2)$$

$$Q_{C_k}^{D_k}(z_{k,i}, x_{k,i}) = \exp \left\{ \sum_{i=1}^k \left(e^{x_{k,i} - z_{k,i}} + e^{z_{k,i+1} - x_{k,i}} \right) + e^{x_{k,k} - z_{k,k}} + e^{-x_{k,k} - z_{k,k}} \right\}. \quad (4.3)$$

One can straightforwardly verify that the following intertwining relations holds

$$H^{D_{k+1}}(x_{k+1,j}) Q_{D_{k+1}}^{D_k}(x_{k+1,j}, x_{k,j}, \lambda_{k+1}) = Q_{D_{k+1}}^{D_k}(x_{k+1,j}, x_{k,j}, \lambda_{k+1}) (H^{D_k}(x_{k,j}) + \frac{1}{2}\lambda_{k+1}^2),$$

where quadratic Hamiltonian operators are given by

$$H^{D_k}(x_{k,j}) = -\frac{1}{2} \sum_{j=1}^k \frac{\partial^2}{\partial x_{k,j}^2} + \sum_{i=1}^{k-1} e^{x_{k,j+1} - x_{k,j}} + e^{-x_{k,k} - x_{k,k-1}}. \quad (4.4)$$

Consider the wave function for D_n Toda chain satisfying the eigenfunction equation

$$H^{D_n}(x) \Psi_{\lambda_1, \dots, \lambda_n}^{D_n}(x_1, \dots, x_n) = \frac{1}{2} \left(\sum_{i=1}^n \lambda_i^2 \right) \Psi_{\lambda_1, \dots, \lambda_n}^{D_n}(x_1, \dots, x_n). \quad (4.5)$$

Then it can be represented in the following integral from

$$\Psi_{\lambda_1, \dots, \lambda_n}^{D_n}(x_1, \dots, x_n) = \int \bigwedge_{k=1}^{n-1} \bigwedge_{i=1}^k dx_{k,i} e^{i\lambda_1 x_{11}} \prod_{k=1}^{n-1} Q_{D_{k+1}}^{D_k}(x_{k+1,j}, x_{k,j}, \lambda_{k+1}), \quad (4.6)$$

where $x_i := x_{n,i}$.

Let us note that there is another form of the integral representation for the kernel of the recursive operator (4.1)

$$\begin{aligned}
Q_{D_{k+1}}^{D_k}(x_{k+1,j}, x_{k,j}, \lambda_{k+1}) &= \frac{1}{\Gamma(2\ell \lambda_{k+1})} \int \bigwedge_{i=1}^k dz_{k,i} dy_k \times \\
&\times \exp \left\{ i\lambda_{k+1} \left(\sum_{i=1}^{k+1} x_{k+1,i} + \sum_{i=1}^k x_{k,i} - 2 \sum_{i=1}^k z_{k,i} + 2y_{k+1} \right) \right\} \times \\
&\times Q_{D_{k+1}}^{C_k}(x_{k+1,1}, \dots, x_{k+1,k+1}; z_{k,1}, \dots, z_{k,k}; y_k) Q_{C_k}^{D_k}(z_{k,1}, \dots, z_{k,k}; x_{k,1}, \dots, x_{k,k}; y_k).
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
Q_{D_{k+1}}^{C_k}(x_{k+1,i}; z_{k,i}, y_k) &= \exp \left\{ \sum_{i=1}^k \left(e^{z_{k,i} - x_{k+1,i}} + e^{x_{k+1,i+1} - z_{k,i}} \right) + e^{-x_{k+1,k+1} - z_{k,k}} + e^{-x_{k+1,k+1} - y_k} \right\}, \\
Q_{C_k}^{D_k}(z_{k,i}; x_{k,i}, y_k) &= \exp \left\{ \sum_{i=1}^k \left(e^{x_{k,i} - z_{k,i}} + e^{z_{k,i+1} - x_{k,i}} \right) + e^{x_{k,k} - z_{k,k}} + e^{-x_{k,k} - z_{k,k}} + e^{-x_{k,k} - y_k} \right\}.
\end{aligned}$$

This integral representation can be easily described in terms of the appropriate Givental diagram. The detailed discussion of the relation with the geometry of D_n flag spaces and the results for other classical finite and affine Lie algebras will be presented in [GLO1].

5 Appendix: Wave function for D_2

Open Toda chain for the root system D_2 can be represented as a couple of non-interacting A_1 Toda chains. This is a consequence of the isomorphism $D_2 = A_1 \oplus A_1$. Quadratic Hamiltonian operator of D_2 Toda chain is given by

$$H^{D_2}(x) = -\frac{1}{2} \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2} + e^{x_{22} - x_{21}} + e^{-x_{22} - x_{21}} \tag{5.1}$$

Changing the variables

$$\xi = x_{22} - x_{21}, \quad \eta = x_{22} + x_{21}, \tag{5.2}$$

one transforms H^{D_2} into the sum of two A_1 Hamiltonians:

$$H^{D_2}(x) = 2 H^{A_1}(\xi) + 2 H^{A_1}(\eta) := 2 \left(-\frac{1}{2} \frac{\partial^2}{\partial \xi^2} + \frac{1}{2} e^\xi \right) + 2 \left(-\frac{1}{2} \frac{\partial^2}{\partial \eta^2} + \frac{1}{2} e^{-\eta} \right). \tag{5.3}$$

The additional quartic D_2 Hamiltonian commuting with (5.1) is given by $H_4^{D_2}(x_1, x_2) = H^{A_1}(\xi) H^{A_1}(\eta)$. Let $\Psi_{\lambda_1, \lambda_2}(x_{21}, x_{22})$ be an eigenfunction of the quadratic D_2 Hamiltonian

$$H^{D_2}(x) \Psi_{\lambda_1, \lambda_2}^{D_2}(x) = \left(\frac{1}{2} \lambda_1^2 + \frac{1}{2} \lambda_2^2 \right) \Psi_{\lambda_1, \lambda_2}^{D_2}(x) \tag{5.4}$$

Taking into account (5.3) one can represent the eigenfunction as a product of two eigenfunctions of A_1 Hamiltonians as

$$\Psi_{\lambda_1, \lambda_2}(x_{21}, x_{22}) = \chi_{\frac{\lambda_2 - \lambda_1}{2}}(-x_{22} - x_{21}) \chi_{\frac{\lambda_2 + \lambda_1}{2}}(x_{22} - x_{21}), \quad (5.5)$$

where $\chi_\nu(y)$ satisfies the equation

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial y^2} + \frac{1}{2} e^y\right) \chi_\nu(y) = \frac{1}{2} \nu^2 \chi_\nu(y). \quad (5.6)$$

Solutions of (5.6) can be expressed through Macdonald function as follows

$$\chi_\nu(y) = K_{2\nu}(2e^y) = -\frac{e^{2\nu}}{2} \int_{-\infty + i\pi}^{\infty + i\pi} e^{2\nu(x+y)} \exp\{e^{-x} + e^{x+2y}\} dx. \quad (5.7)$$

Below we explicitly demonstrate the factorisation (5.5) using the integral representation for D_2 Toda chain wave function described in the previous Section.

The eigenfunction of H^{D_2} has the following integral representation (see (4.6) for $n = 2$)

$$\begin{aligned} \Psi_{\lambda_1, \lambda_2}^{D_2}(x_{21}, x_{22}) &= \int \int dx_{11} dz_{11} dy \exp\left\{i\lambda_2(x_{22} + x_{21}) - i2\lambda_2(z_{11} - y) + i(\lambda_2 + \lambda_1)x_{11}\right\} \times \\ &\times \exp\left\{e^{z_{11} - x_{21}} + e^{x_{22} - z_{11}} + e^{-x_{22} - z_{11}} + e^{x_{11} - z_{11}} + e^{-x_{11} - z_{11}} + e^{-x_{22} - y} + e^{-x_{11} - y}\right\}. \end{aligned} \quad (5.8)$$

One can integrate explicitly over y to get the following representation

$$\begin{aligned} \Psi_{\lambda_1, \lambda_2}^{D_2}(x_{21}, x_{22}) &= \Gamma(2i\lambda_2) \int \int dx_{11} dz_{11} (e^{-x_{22}} + e^{-x_{11}})^{i2\lambda_2} \times \\ &\times \exp\left\{i\lambda_2(x_{22} + x_{21}) - i2\lambda_2 z_{11} + i(\lambda_2 + \lambda_1)x_{11}\right\} \times \\ &\times \exp\left\{e^{z_{11} - x_{21}} + e^{x_{22} - z_{11}} + e^{-x_{22} - z_{11}} + e^{x_{11} - z_{11}} + e^{-x_{11} - z_{11}}\right\}. \end{aligned} \quad (5.9)$$

Let us introduce new variables

$$\xi = x_{22} - x_{21}, \quad \eta = x_{22} + x_{21}, \quad z'_{11} = z_{11} - \frac{x_{22} + x_{21}}{2}, \quad x'_{11} = x_{11} - \frac{x_{22} + x_{21}}{2}. \quad (5.10)$$

Then

$$\begin{aligned} \Psi_{\lambda_1, \lambda_2}^{D_2}(x_{21}, x_{22}) &= \Gamma(2i\lambda_2) \int \int dx'_{11} dz'_{11} (e^{-x_{22}} + e^{-x'_{11} + \frac{\eta}{2}})^{i2\lambda_2} \times \\ &\times \exp\left\{-i\frac{(\lambda_2 + \lambda_1)}{2}\eta - i2\lambda_2 z'_{11} + i(\lambda_2 + \lambda_1)x'_{11}\right\} \times \\ &\times \exp\left\{e^{z'_{11} + \frac{\xi}{2}} + e^{\frac{\xi}{2} - z'_{11}} + e^{x'_{11} - z'_{11}} + e^{-\frac{\xi}{2} - \eta - z'_{11}} + e^{-x'_{11} - z'_{11} - \eta}\right\}. \end{aligned} \quad (5.11)$$

Define new variable as follows

$$e^\tau = e^{-z'_{11} - \eta} (e^{-\frac{\xi}{2}} + e^{-x'_{11}}), \quad (5.12)$$

$$e^t = e^{x'_{11} - z'_{11} - \frac{\xi}{2} - \eta} \left(e^{-\frac{\xi}{2}} + e^{-x'_{11}} \right). \quad (5.13)$$

As a result we obtain the following integral representation

$$\begin{aligned} \Psi_{\lambda_1, \lambda_2}^{D_2}(x_{21}, x_{22}) &= \Gamma(2i\lambda_2) \int \int d\tau dt \exp \left\{ -i \frac{(\lambda_2 - \lambda_1)}{2} \eta + i(\lambda_2 - \lambda_1) + \right. \\ &\quad \left. + i \frac{(\lambda_2 + \lambda_1)}{2} \xi + i(\lambda_2 + \lambda_1)t \right\} \exp \left\{ e^{-\tau} + e^{\tau - \eta} + e^{-t} + e^{t + \xi} \right\} = \\ &= 4e^{-2\pi\lambda_2} \Gamma(2i\lambda_2) K_{i\frac{\lambda_2 + \lambda_1}{2}}(2e^{\frac{\xi}{2}}) K_{i\frac{\lambda_2 - \lambda_1}{2}}(2e^{-\frac{\eta}{2}}). \end{aligned} \quad (5.14)$$

Thus we have demonstrated that the proposed integral representation for D_n Toda chain wave functions is compatible with the factorization of D_2 Toda chain.

References

- [AvM] M. Adler, P. van Moerbeke, *Toda Darboux maps and vertex operators*, Internat. Math. Res. Notices, **10** (1998) 489-511.
- [B] R.J. Baxter, *Exactly solved models in statistical mechanics*, London: Academic Press, 1982.
- [DS] V.G. Drinfel'd, V.V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, **v. 24** (1984), 81-180.
- [GKLO] A. Gerasimov, S. Kharchev, D. Lebedev, S. Oblezin, *On a Gauss-Givental representation for quantum Toda chain wave function*, To be published in Int. Math. Res. Notices, (2006), [arXiv:math.RT/0505310].
- [GLO] A. Gerasimov, D. Lebedev, S. Oblezin, *Givental Integral Representation for Classical Groups*, [arXiv:math.RT/0608152].
- [GLO1] A. Gerasimov, D. Lebedev, S. Oblezin, *On a Gauss-Givental representation for classical groups*, to appear.
- [Gi] A. Givental, *Stationary Phase Integrals, Quantum Toda Lattices, Flag Manifolds and the Mirror Conjecture*, AMS Trans. (2) **180** (1997), 103-115.
- [K] V. Kac, *Infinite-dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge 1990.
- [PG] V. Pasquier, M. Gaudin, *The periodic Toda chain and a matrix generalization of the Bessel function recursion relation*, J. Phys. A **25** (1992), 5243-5252.
- [RSTS] A.G. Reyman, M.A. Semenov-Tian-Shansky, *Integrable Systems. Group theory approach*, Modern Mathematics, Moscow-Igevsk: Institute Computer Sciences, 2003.
- [V] A. P. Veselov, *Integrable maps*, Russian Math. Surveys **46:5** (1991) 1-51.